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Lie series in an extended region of phase space

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Abstract. A perturbative approach has been worked out that explicitly gives the canonical transformations which solve a certain class of analytic symplectic maps. The advantage of this method is that the perturbation series need not be derived with respect to the origin in phase space. Rather it is computed about a given radius while also circumventing the familiar problem of 'small denominators'. The method is applied to a simple system in one degree of freedom for which computer output shows remarkable convergence near a dominant resonance.

1. Introduction

The notation and formalism used here is essentially that of Dragt and Finn [1, 2] and Dragt [3] but is also due in part to Forest [4, 5, 6]. Forest has written a set of computer programs that are a Lie algebraic implementation of the differential algebra (DA) package of Berz [7]. These programs provide a novel way of working with Lie maps by manipulating them in Taylor series form. Presently, the perturbative methods used in these programs proceed relative to some fixed point of the map, most often the origin in phase space. Unfortunately, this method at best converges up to the separatrix of the nearest resonant island chain it sees. However, the KAM theorem assues us that for sufficiently small nonlinearity invariant tori will exist throughout the phase space.

My purpose has been to find an approach that instead of relying on a fixed point, converges at best at some specified radial distance from the origin. Strictly speaking none of these asymptotic series converge. Instead for near-integrable systems the approximation keeps getting better up to a certain order in the expansion after which additional terms make the series diverge. For simplicity the examples given below are in one degree of freedom although they are very easily generalised to many dimensions.

2. Notation and preliminaries

$:f:=$ Poisson bracket operator

$$:f:g=[f,g]=\frac{\partial f}{\partial x}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial x}$$

where f and g are functions of the phase space variables x and p.

$$\exp(:f:)g=g+[f,g]+[f,[f,g]]/2!+\ldots.$$

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We define J and ϕ so that

$$x = \sqrt{2J}\sin\phi$$
$$p = \sqrt{2J}\cos\phi.$$

We begin with a map of the form:

$$M = \exp(:-\mu J:)\exp(:\alpha f:) \tag{1}$$

where μ is the constant tune, α is a smallness parameter, and f is a polynomial in x and p of higher than second order. More explicitly, for the case $f = x^4/4$, M maps the initial point (x, p) into the final point (X, P) as shown:

$$\begin{aligned} X &= x\cos\mu + p\sin\mu \\ P &= -x\sin\mu + p\cos\mu + X^3. \end{aligned} \tag{2}$$

(2)

We will be going back to this particular map as a specific example later on. For now notice that the map $\exp(:-\mu J:)$ simply acts like a rotation in phase space by an angle μ. For brevity we will denote this map by R. It can be shown that all maps of the form (1) are symplectic [8] whereas the map (2) might describe a harmonic oscillator experiencing an octupole kick. Our goal is to find a similarity transformation linking M with another map N that depends only on the radial variable J.

$$M = A^{-1}NA \tag{3}$$

where N is of the form: $R\exp(:k(J):)$. Once this is done the invariant curves of the map M are just given by the function:

$$J^* = A^{-1}J = \text{constant}$$

since

$$\begin{aligned} MJ^* &= A^{-1}NAA^{-1}J \\ &= A^{-1}NJ \\ &= A^{-1}J \end{aligned}$$

and because the Poisson bracket of J with itself is zero.

Before we can proceed any further we need a few more tools.

Theorem 1. Let f and g be arbitrary functions of the phase space variables, then

$$e^{:f:}e^{:g:}e^{:-f:} = \exp(:e^{:f:}g:).$$

Theorem 2. The Cambell-Baker-Hausdorff formula [9]. Let A, B and C be non-commuting operators, and α a smallness parameter, together satisfying

$$\exp(\alpha A)\exp(\alpha B) = \exp(\alpha C)$$

then

$$C = A + B + \alpha\{A, B\}/2 + \alpha^2\{A, \{A, B\}\}/12 + \alpha^2\{B, \{B, A\}\}/12 + \ldots$$

where $\{A, B\} = AB - BA$ is the commutator of A and B.

Theorem 3. Let f and g be arbitrary functions of the phase space variables, then [10]:

$$\{:f:, :g:\} = :[f, g]:.$$

Combining theorems 2 and 3 provides a way of manipulating Lie maps as infinite series; all one really needs is the ability to compute Poisson brackets.

3. Second-order factorisation

We are now ready to find the maps N and A^{-1}. I shall first present a second-order factorisation about the origin and then compare this with a perturbation series that is displaced from the origin.

Using (3) we begin as follows.

$$N = AMA^{-1}$$

$$N = \exp(:\alpha F + \alpha^2 G:) R \exp(:\alpha f:) \exp(:-\alpha F - \alpha^2 G:)$$

$$N = RR^{-1} \exp(:\alpha F + \alpha^2 G:) R \exp(:\alpha f:) \exp(:-\alpha F - \alpha^2 G:).$$

Applying theorem 1 gives:

$$N = R \exp(:\alpha R^{-1} F + \alpha^2 R^{-1} G:) \exp(:\alpha f:) \exp(:-\alpha F - \alpha^2 G:).$$

Now use the CBH formula to combine the exponents to second order in α:

$$N = R \exp(:\alpha(f-(1-R^{-1})F) + \alpha^2([R^{-1}F, f]/2 + [F, f]/2 + [F, R^{-1}F]/2 - (1-R^{-1})G):). \tag{4}$$

It is up to us to choose the functions F and G so that N is a function of J alone. First break f into components $f_\perp$ and $f_\parallel$ so that:

$$f(J, \phi) = f_\perp(J) + f_\parallel(J, \phi).$$

Choose

$$F = (1-R^{-1})^{-1} f_\parallel$$

$$G = (1-R^{-1})^{-1}([R^{-1}F, f] + [F, f] + [F, R^{-1}F])_\parallel/2.$$

The reason for projecting out the perpendicular components of the functions above is that the operator $(1-R^{-1})^{-1}$ is undefined when acting on terms that depend only on J. This naturally suits our purpose since we do not wish to cancel these terms.

Example.

$$f = x^4/4 \Rightarrow f_\perp = \tfrac{3}{8}J^2 \qquad f_\parallel = J^2(\cos 4\phi - 4\cos 2\phi)/8$$

$$F = (1-e^{:\mu J:})^{-1} J^2(\cos 4\phi - 4\cos 2\phi)/8$$

$$F = \frac{J^2}{16}\left(\cos 4\phi + \frac{\sin 4\phi \sin 4\mu}{1-\cos 4\mu} - 4\cos 2\phi - \frac{4\sin 2\phi \sin 2\mu}{1-\cos 2\mu}\right).$$

Thus to first order:

$$N = R \exp(:\alpha 3J^2/8:).$$

The factor $\sin m\mu/(1-\cos m\mu)$ is ever present in this perturbation series and can become quite large when $m\mu$ is near an integral multiple of 2π. The basic idea behind what I am about to do differently is that we need not form our perturbation series about the unperturbed tune μ. Rather shifting the tune to some value μ' displaces us away from the origin and possible resonant island chains. To see how this works we return to the map (1) and insert the identity map twice:

$$M = \exp(:-J_0\phi:) \exp(:J_0\phi:) \exp(:-\mu J:) \exp(:\alpha f(J, \phi):) \exp(:-J_0\phi:) \exp(:J_0\phi:)$$

$$M = \exp(:-J_0\phi:) \exp(:-\mu J:) \exp(:\alpha f(J+J_0, \phi):) \exp(:+J_0\phi:). \tag{5}$$

Whereas R was a rotation in phase space, the map $\exp(:J_0\phi:)$ provides a displacement in the variable J by an amount J_0. Previously, we stipulated that the function f be a polynomial in x and p of higher than second degree. The inclusion of J_0 in f has undoubtedly generated a linear dependence of f on J. We wish to factor out this linear part.

$$\exp(:\alpha f'(J+J_0, \phi, a):) = \exp(:\alpha a J:) \exp(:\alpha f(J+J_0, \phi):). \tag{6}$$

We choose 'a' so that the linear part of f' vanishes. This gives:

$$\begin{aligned} M &= \exp(:-J_0\phi:) \exp(:-(\mu+\alpha a)J:) \exp(:\alpha f'(J+J_0, J_0, \phi):) \exp(:+J_0\phi:) \\ M &= R' \exp(:\alpha f'(J, J_0, \phi):). \end{aligned} \tag{7}$$

At this point we can use Forest's program to factorise this map in the usual way. The parameter J_0 can be seen either as a spatial displacement in phase space or as a way of avoiding the small denominators associated with a highly resonant μ.

4. A specific example

Let us now work out as a specific example the map (2). We shall go to second order in the smallness parameter α.

$$\begin{aligned} M &= \exp(:-\mu J:) \exp(:\alpha x^4/4:) \\ M &= R \exp(:\alpha J^2 \sin^4\phi:). \end{aligned}$$

After introducing the identity we get

$$\begin{aligned} M &= \exp(:-J_0\phi:) \exp(:-\mu J:) \exp(:\alpha(J+J_0)^2 \sin^4\phi:) \exp(:+J_0\phi:) \\ &\quad \times \exp(:\alpha(J+J_0)^2 \sin^4\phi:) = \exp(:-\alpha a J:) \exp(:\alpha f':) \\ f' &= aJ + (J+J_0)^2(\cos 4\phi - 4\cos 2\phi) + \tfrac{3}{8}(J+J_0)^2 \\ &\quad + \alpha a(J+J_0)^2(\sin 4\phi - 2\sin 2\phi) + O(\alpha^2). \end{aligned} \tag{8}$$

Equating the linear part of f' to zero in (8) gives $a = -\frac{3}{4}J_0$.

As in equation (7) we are left with the map

$$M' = \exp(:-\mu' J:) \exp(:\alpha g + \alpha^2 h:)$$

where $\mu' = \mu - \alpha\frac{3}{4}J_0$

$$g = J^2 \sin^4\phi - \tfrac{3}{4}JJ_0 \tag{9}$$

and $h = \frac{3}{16}J_0J^2(2\sin 2\phi - \sin 4\phi)$.

We can compare what factorisation of the second-order map M' gives with what Forest's program would give for the map S below:

$$S = \exp(:-\mu' J:) \exp(:\alpha J^2 \sin^4\phi:).$$

They are nearly the same except that we are required to include an additional term in equation (4) namely

$$\begin{aligned} &\alpha^2(1-R^{-1})^{-1}(h + [R^{-1}F + F, -\tfrac{3}{4}J_0J]/2) \\ &\quad = \alpha^2(1-R^{-1})^{-1}\tfrac{3}{16}J^2J_0\left(2\sin 2\phi + \frac{2\cos 2\phi \sin 2\mu'}{1-\cos 2\mu'}\right. \\ &\qquad \left. - \sin 4\phi - \frac{\cos 4\phi \sin 4\mu'}{1-\cos 4\mu'}\right) \\ &\quad = \alpha^2\tfrac{3}{2}J_0J^2(2(1+a^2)\sin 2\phi - (1+b^2)\sin 4\phi) \end{aligned}$$

where $a = \sin 2\mu'/(1-\cos 2\mu')$ and $b = \sin 4\mu'/(1-\cos 4\mu')$.

5. Results

If we look at the effect of applying the transformation A^{-1} to particles tracked through M we see that the invariant tori in the original space are transformed into circles in the new space. (See the appendix for the explicit derivation.) This provides a visual way of testing the perturbation series using a computer. Figure 1(a) shows the x-p phase portrait of the map (2) for $\mu = 0.255\,(2\pi)$ and at various initial conditions. Because of the dependence of f on x^4 this map is highly sensitive to the resonance at $\mu/2\pi = \frac{1}{4}$. These points are then mapped through the second-order transformation A^{-1} calculated above and the result is shown in figure 1(b). The radial parameter $J_0 = 0.4$ was chosen in order to displace us outside the large, period four-island chain. The same procedure is followed in figure 2 except that for this map $\mu = 0.245(2\pi)$. The reasoning behind displacing the tune is simply that if we are interested in the behaviour of the mapping away from the origin then that is where we must base our series.

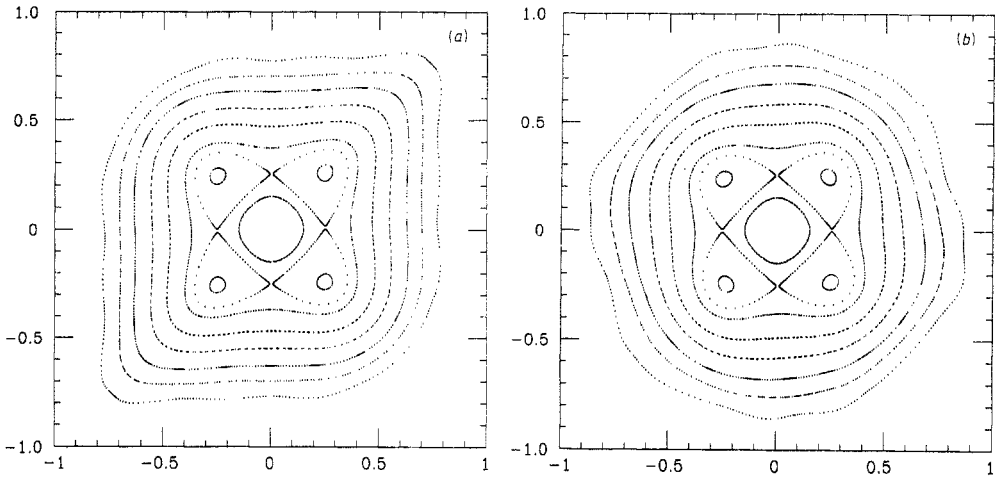

Figure 1. Trajectories of the mapping (2): (a) with $\mu = 0.255\,(2\pi)$; (b) transformed using the operator A^{-1} with $J_0 = 0.4$.

In figures 3 and 4 we can see what happens when we set the parameter $J_0 = 0$ in the perturbation series. In figure 3(b) tori are transformed into circles all the way out to the separatrix of the resonant island chain. However, further out and beyond, this series is seen to diverge wildly. Although there is no large island chain near the origin in figure 4, the resonant denominator $\sin 4\mu/(1-\cos 4\mu)$ still causes the series to diverge further out.

6. Generalise with the DA package

It turned out to be rather straightforward for Forest to integrate this new approach into his Lie algebra programs. Because these programs rely on the differential algebra (DA) package of Berz one can go to arbitrary order in this type of perturbation series without ever having to use the CBH formula; exponential maps are concatenated in Taylor series form. The method works in two, four and six dimensions but there is a drawback to its current implementation which I show how to correct below.

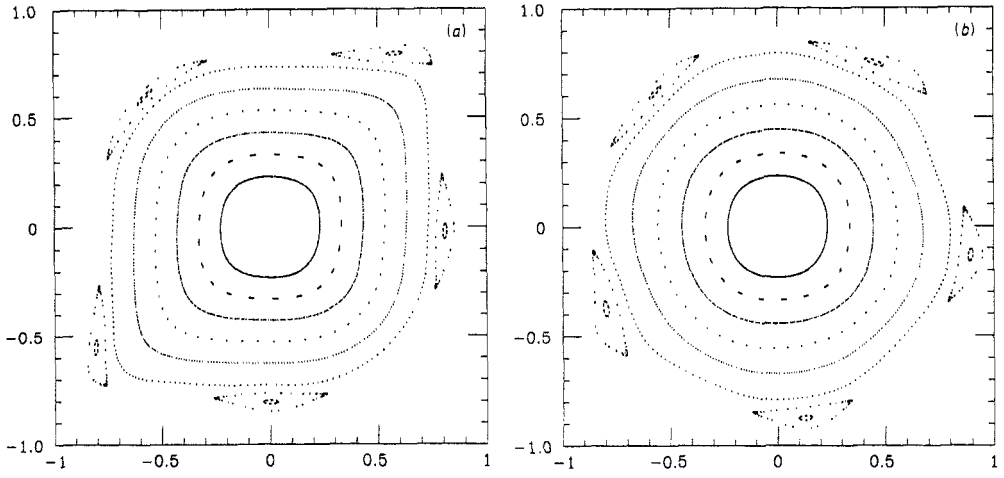

Figure 2. Trajectories of the mapping (2): (*a*) with $\mu = 0.245\ (2\pi)$; (*b*) transformed using the operator A^{-1} with $J_0 = 0.4$.

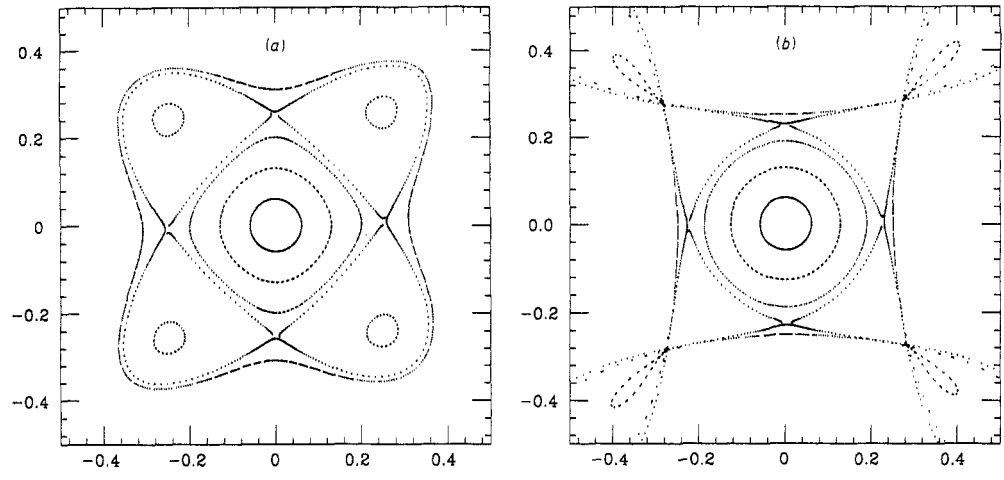

Figure 3. Trajectories of the mapping (2): (*a*) with $\mu = 0.255\ (2\pi)$; (*b*) transformed using the operator A^{-1} with $J_0 = 0$.

Basically what Forest† did was allow the user to select the tune μ' about which the series would be formed. This way the series would converge best for the curve having tune μ' regardless of resonances near the origin in phase space. Unfortunately selecting a specific radial distance requires advance knowledge of how the tune varies with amplitude at high order in the perturbation series.

In any case, the ability to go to very high order in the expansion shows undeniably that shifting the tune produces a series that converges away from the origin. Figure 5(*a*) is a tenth-order ($\propto x^{10}$) extension of figure 1(*b*). Here the shifted tune μ' was chosen to make the $m = 4$ resonant denominators small thereby displacing us outside

† Forest felt that the modifications to his existing programs were so slight that they did not require publication.

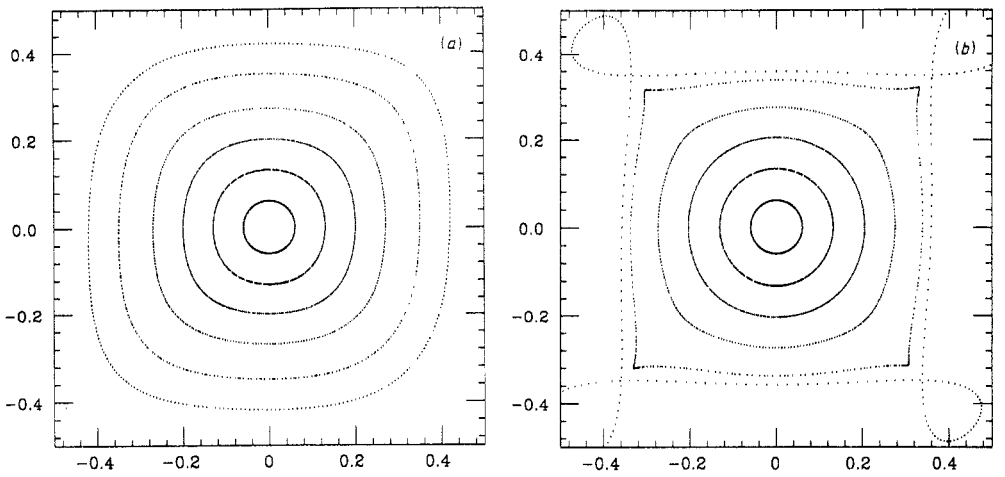

Figure 4. Trajectories of the mapping (2): (*a*) with $\mu = 0.245\,(2\pi)$; (*b*) transformed using the operator A^{-1} with $J_0 = 0$.

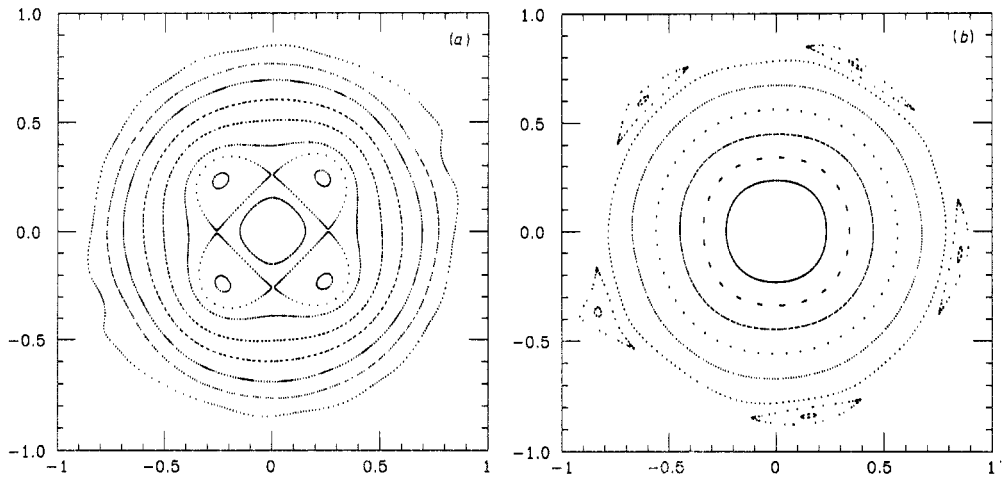

Figure 5. (*a*) The trajectories of figure 1(*a*) transformed by the tenth-order map A^{-1} with $\mu' = 0.207\,(2\pi)$. (*b*) The trajectories of figure 2(*a*) transformed by the tenth-order map A^{-1} with $\mu' = 0.207\,(2\pi)$.

of the island chain. Figure 5(*b*) is the same as figure 2(*b*) except that here the series was also carried to tenth order.

The second-order calculation tells us approximately what tune to pick to be at a certain radial value of J_0. However, as the order of the series is increased, cross terms at high order move us away from the selected value of J_0. To make a truly 'fixed J_0' perturbation series rather than a 'fixed tune' series requires an iterative procedure. Instead of selecting μ' only at the beginning of the calculation, at every order we chose a different tune so as to drag ourselves back onto the curve at J_0. The idea of selecting multiple tunes about which to form our perturbation series is a natural extension of choosing a tune different from that at the origin.

7. A multiple frequency expansion

To illustrate this idea I present a second-order calculation similar to that above except with a different tune at every order in the expansion. The method is quite easily extended to higher order and more dimensions.

Again begin with

$$M = \exp(:-\mu J:)\exp(:\alpha f:) \qquad f = f_\perp(J) + f_\parallel(J,\phi)$$

$$N = AMA^{-1}$$

$$= \exp(:\alpha^2 G:)\exp(:\alpha F:)\exp(:-\mu J:)\exp(:-\alpha a J:)\exp(:\alpha a J:)\exp(:\alpha f:)$$
$$\times \exp(:-\alpha F:)\exp(:-\alpha^2 G:).$$

Choose $a = -(\partial f_\perp/\partial J)_{J_0}$

$$N = \exp(:\alpha^2 G:)\exp(:\alpha F:)\exp(:-(\mu+\alpha a)J:)\exp(:\alpha a J + \alpha f + \tfrac{1}{2}a\alpha^2[J, f_\parallel]:)$$
$$\times \exp(:-\alpha F:)\exp(:-\alpha^2 G:).$$

In the earlier calculation J was actively translated by the amount J_0 and 'a' was chosen in order to cancel the linear part in f. The step above is equivalent to this but gives a lesser feeling of translating in space.

Define $\mu' = \mu + \alpha a$

$$F = (1 - \exp(\mu' J))^{-1} f_\parallel.$$

At first order

$$N = \exp(:-(\mu+\alpha a)J:)\exp(:\alpha a J + \alpha f_\perp:)$$

$$\text{tune} = \mu - \alpha(\partial f_\perp/\partial J).$$

At J_0 tune $= \mu + \alpha a$.

Continuing on to second order:

$$N = \exp(:\alpha^2 G:)R'\exp(:\alpha a J + \alpha f_\perp + (a\alpha^2/2)[J, f_\parallel] + (a\alpha^2/2)[F, J]$$
$$+ (\alpha^2/2)[R'^{-1}F, f - F] + (\alpha^2/2)[F, f]:)\exp(:-\alpha^2 G:).$$

Define g so that

$$N = \exp(:\alpha^2 G:)\exp(:-\mu' J:)\exp(:-\alpha^2 b J:)\exp(:\alpha^2 b J:)\exp(:\alpha a J + \alpha f_\perp + \alpha^2 g:)$$
$$\times \exp(:-\alpha^2 G:).$$

Choose $b = -(\partial g_\perp/\partial J)_{J_0}$.

Define $\mu'' = \mu' + \alpha^2 b$

$$G = (1 - \exp(\mu'' J))^{-1} g_\parallel.$$

At second order

$$N = \exp(:-(\mu + \alpha a + \alpha^2 b)J:)\exp(:\alpha a J + \alpha f_\perp + \alpha^2 b J + \alpha^2 g_\perp:)$$
$$= \exp(:-\mu J + \alpha f_\perp + \alpha^2 g_\perp:)$$

$$\text{tune} = \mu - \alpha(\partial f_\perp/\partial J) - \alpha^2(\partial g_\perp/\partial J).$$

At J_0 tune $= \mu + \alpha a + \alpha^2 b$.

This process repeats to all orders and only depends on our initial choice of J_0. Notice that at every order in the calculation we are working with the tune at J_0.

8. Conclusion

It has been shown that by using the standard techniques of Lie perturbation theory one can construct the canonical transformations which solve a certain class of analytic symplectic maps in a previously inaccessible region of phase space. The most direct application of these techniques is in the field of accelerator physics where for large complicated machines the mapping approach is supremely advantageous over Hamiltonian methods. A potential machine is the proposed SSC for which the question of long term stability is very important. Also relevant to the subject of computer oriented perturbation methods is the work of Warnock and Ruth [12] who succeeded in approximating invariant curves outside of island chains by direct substitution of the Fourier series into the Hamilton-Jacobi equation.

Acknowledgments

The author wishes to thank D Judd for supporting this work and also E Forest for both suggesting the original problem and for the use of his computer programs. A final word should go to M Berz whose DA package was essential for the high-order calculations.

Appendix

Let z be our coordinate vector and let J_i and ϕ_i be our starting point in phase space:

$$z_i = (J_i, \phi_i).$$

Let $\xi = M(z_i)z_i$ then

$$\begin{aligned} z_f &= A^{-1}(\xi)M(z_i)z_i \\ &= A^{-1}(M(z_i)z_i)M(z_i)z_i \\ &= M(z_i)A^{-1}(z_i)M^{-1}(z_i)M(z_i)z_i \\ &= A^{-1}(z_i)N(z_i)z_i \\ &= (A^{-1}J_i, A^{-1}N\phi_i). \end{aligned}$$

If N is of the form $R\exp(:k(J):)$ then:

$$z_f = A^{-1}(J_i, \phi_i + \mu - (\partial k/\partial J_i))$$

and after n iterations:

$$z^n = A^{-1}(J_i, \phi_i + n\mu - n(\partial k/\partial J_i)).$$

So the transformed J is a constant in time while ϕ increases at a constant rate.

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